CHAPTER 2
POSITIVE INTEGERS

The purpose of this chapter is to review the methods of combining integers. We have already used one combination process in our discussion of counting. We will extend the idea of counting, which is nothing more than simple addition, to develop a systematic method for adding numbers of any size. We will also learn the meaning of subtraction, multiplication, and division.

ADDITION AND SUBTRACTION

In the following discussion, it is assumed that the reader knows the basic addition and subtraction tables, which present such facts as the following: $2 + 3 = 5$, $9 + 8 = 17$, $8 - 3 = 5$, etc.

The operation of addition is indicated by a plus sign (+) as in $8 + 4 = 12$. The numbers 8 and 4 are ADDENDS and the answer (12) is their SUM. The operation of subtraction is indicated by a minus sign (-) as in $9 - 3 = 6$. The number 9 is the MINUEND, 3 is the SUBTRAHEND, and the answer (6) is their DIFFERENCE.

REGROUPING

Addition may be performed with the addends arranged horizontally, if they are small enough and not too numerous. However, the most common method of arranging the addends is to place them in vertical columns. In this arrangement, the units digits of all the addends are alined vertically, as are the tens digits, the hundreds digits, etc. The following example shows three addends arranged properly for addition:

$$\begin{array}{r} 357 \\ 1{,}845 \\ \underline{22} \end{array}$$

It is customary to draw a line below the last addend, placing the answer below this line. Subtraction problems are arranged in columns in the same manner as for addition, with a line at the bottom and the answer below this line.

Carry and Borrow

Problems involving several addends, with two or more digits each, usually produce sums in one or more of the columns which are greater than 9. For example, suppose that we perform the following addition:

$$\begin{array}{r} 357 \\ 845 \\ \underline{22} \\ 1{,}224 \end{array}$$

The answer was found by a process called "carrying." In this process extra digits, generated when a column sum exceeds 9, are carried to the next column to the left and treated as addends in that column. Carrying may be explained by grouping the original addends. For example, 357 actually means 3 hundreds plus 5 tens plus 7 units. Rewriting the problem with each addend grouped in terms of units, tens, etc., we would have the following:

$$\begin{array}{rrr} 300 + & 50 + & 7 \\ 800 + & 40 + & 5 \\ \hline & 20 + & 2 \\ \hline 1{,}100 + & 110 + & 14 \end{array}$$

The "extra" digit in the units column of the answer represents 1 ten. We regroup the columns of the answer so that the units column has no digits representing tens, the tens column has no digits representing hundreds, etc., as follows:

$$\begin{aligned} 1{,}100 + 110 + 14 &= 1{,}100 + 110 + 10 + 4 \\ &= 1{,}100 + 120 + 4 \\ &= 1{,}100 + 100 + 20 + 4 \\ &= 1{,}200 + 20 + 4 \\ &= 1{,}000 + 200 + 20 + 4 \\ &= 1{,}224 \end{aligned}$$

When we carry the 10 from the expression $10 + 4$ to the tens column and place it with the 110 to make 120, the result is the same as if we had added 1 to the digits 5, 4, and 2 in the tens column of the original problem. Therefore, the thought process in addition is as follows: Add the 7, 5, and 2 in the units column, getting a sum of 14. Write down the 4 in the units column of the answer and carry the 1 to the tens column. Mentally add the 1 along with the other digits in the tens column, getting a sum of 12. Write down the 2 in the tens column of the answer and carry the 1 to the hundreds column. Mentally add the 1 along with the other digits in the hundreds column, getting a sum of 12. Write down the 2 in the hundreds column of the answer and carry the 1 to the thousands column. If there were other digits in the thousands column to which the 1 could be added, the process would continue as before. Since there are no digits in the thousands column of the original problem, this final 1 is not added to anything, but is simply written in the thousands place in the answer.

The borrow process is the reverse of carrying and is used in subtraction. Borrowing is not necessary in such problems as 46 - 5 and 58 - 53. In the first problem, the thought process may be "5 from 6 is 1 and bring down the 4 to get the difference, 41." In the second problem, the thought process is "3 from 8 is 5" and "5 from 5 is zero," and the answer is 5. More explicitly, the subtraction process in these examples is as follows:

$$\begin{array}{r} 40 + 6 \\ \underline{5} \\ 40 + 1 = 41 \end{array} \qquad \begin{array}{r} 50 + 8 \\ \underline{50 + 3} \\ 0 + 5 = 5 \end{array}$$

This illustrates that we are subtracting units from units and tens from tens.

Now consider the following problem where borrowing is involved:

$$\begin{array}{r} 43 \\ \underline{8} \end{array}$$

If the student uses the borrowing method, he may think "8 from 13 is 5 and bring down 3 to get the difference, 35." In this case what actually was done is as follows:

$$\begin{array}{r} 30 + 13 \\ \underline{8} \\ 30 + 5 = 35 \end{array}$$

A 10 has been borrowed from the tens column and combined with the 3 in the units column to make a number large enough for subtraction of the 8. Notice that borrowing to increase the value of the digit in the units column reduces the value of the digit in the tens column by 1.

Sometimes it is necessary to borrow in more than one column. For example, suppose that we wish to subtract 2,345 from 5,234. Grouping the minuend and subtrahend in units, tens, hundreds, etc., we have the following:

$$\begin{array}{r} 5{,}000 + 200 + 30 + 4 \\ \underline{2{,}000 + 300 + 40 + 5} \end{array}$$

Borrowing a 10 from the 30 in the tens column, we regroup as follows:

$$\begin{array}{r} 5{,}000 + 200 + 20 + 14 \\ \underline{2{,}000 + 300 + 40 + 5} \end{array}$$

The units column is now ready for subtraction. By borrowing from the hundreds column, we can regroup so that subtraction is possible in the tens column, as follows:

$$\begin{array}{r} 5{,}000 + 100 + 120 + 14 \\ \underline{2{,}000 + 300 + 40 + 5} \end{array}$$

In the final regrouping, we borrow from the thousands column to make subtraction possible in the hundreds column, with the following result:

$$\begin{array}{r} 4{,}000 + 1{,}100 + 120 + 14 \\ \underline{2{,}000 + 300 + 40 + 5} \\ 2{,}000 + 800 + 80 + 9 = 2{,}889 \end{array}$$

In actual practice, the borrowing and regrouping are done mentally. The numbers are written in the normal manner, as follows:

$$\begin{array}{r} 5{,}234 \\ \underline{-2{,}345} \\ 2{,}889 \end{array}$$

The following thought process is used: Borrow from the tens column, making the 4 become 14. Subtracting in the units column, 5 from 14 is 9. In the tens column, we now have a 2 in the minuend as a result of the first borrowing operation. Some students find it helpful at first to cancel any digits that are reduced as a result of borrowing, jotting down the digit of next lower value just above the canceled digit. This has been done in the following example:

$$\begin{array}{r} 4\ 12 \\ \cancel{5},\cancel{2}\cancel{3}\cancel{4} \\ \underline{-2,345} \\ 2,889 \end{array}$$

After canceling the 3, we proceed with the subtraction, one column at a time. We borrow from the hundreds column to change the 2 that we now have in the tens column into 12. Subtracting in the tens column, 4 from 12 is 8. Proceeding in the same way for the hundreds column, 3 from 11 is 8. Finally, in the thousands column, 2 from 4 is 2.

Practice problems. In problems 1 through 4, add the indicated numbers. In problems 5 through 8, subtract the lower number from the upper.

1. Add 23, 468, 7, and 9,045.

2. 129	3. 9,497	4. 67,856	
958	6,364	22,851	
787	4,269	44,238	
436	9,785	97,156	
5. 709	6. 8,700	7. 7,928	8. 75,168
594	5,008	5,349	28,089

Answers:

1. 9,543	2. 2,310	3. 29,915	4. 232,101
5. 115	6. 3,692	7. 2,579	8. 47,079

Denominate Numbers

Numbers that have a unit of measure associated with them, such as yard, kilowatt, pound, pint, etc., are called DENOMINATE NUMBERS. The word "denominate" means the numbers have been given a name; they are not just abstract symbols. To add denominate numbers, add all units of the same kind. Simplify the result, if possible. The following example illustrates the addition of 6 ft 8 in. to 4 ft 5 in.:

$$\begin{array}{rr} 6\text{ ft} & 8\text{ in.} \\ \underline{4\text{ ft}} & \underline{5\text{ in.}} \\ 10\text{ ft} & 13\text{ in.} \end{array}$$

Since 13 in. is the equivalent of 1 ft 1 in., we regroup the answer as 11 ft 1 in.

A similar problem would be to add 20 degrees 44 minutes 6 seconds to 13 degrees 22 minutes 5 seconds. This is illustrated as follows:

$$\begin{array}{rrr} 20\text{ deg} & 44\text{ min} & 6\text{ sec} \\ \underline{13\text{ deg}} & \underline{22\text{ min}} & \underline{5\text{ sec}} \\ 33\text{ deg} & 66\text{ min} & 11\text{ sec} \end{array}$$

This answer is regrouped as 34 deg 6 min 11 sec.

Numbers must be expressed in units of the same kind, in order to be combined. For instance, the sum of 6 kilowatts plus 1 watt is not 7 kilowatts nor is it 7 watts. The sum can only be indicated (rather than performing the operation) unless some method is used to write these numbers in units of the same value.

Subtraction of denominate numbers also involves the regrouping idea. If we wish to subtract 16 deg 8 min 2 sec from 28 deg 4 min 3 sec, for example, we would have the following arrangement:

$$\begin{array}{rrr} 28\text{ deg} & 4\text{ min} & 3\text{ sec} \\ \underline{-16\text{ deg}} & \underline{8\text{ min}} & \underline{2\text{ sec}} \end{array}$$

In order to subtract 8 min from 4 min we regroup as follows:

$$\begin{array}{rrr} 27\text{ deg} & 64\text{ min} & 3\text{ sec} \\ \underline{-16\text{ deg}} & \underline{8\text{ min}} & \underline{2\text{ sec}} \\ 11\text{ deg} & 56\text{ min} & 1\text{ sec} \end{array}$$

Practice problems. In problems 1, 2, and 3 add. In problems 4, 5, and 6 subtract the lower number from the upper.

1. 6 yd 2 ft 7 in.	4. 15 hr 25 min 10 sec
1 ft 9 in.	6 hr 50 min 35 sec
2 yd 10 in.	
2. 9 hr 47 min 51 sec	5. 125 deg
3 hr 36 min 23 sec	47 deg 9 min 14 sec
5 hr 15 min 23 sec	
3. 10 wks 5 days 7 hrs	6. 20 wks 2 days 10 hrs
22 wks 3 days 10 hrs	7 wks 6 days 15 hrs
3 wks 4 days 12 hrs	

Answers:

1. 9 yd 2 ft 2 in.
2. 18 hr 39 min 37 sec
3. 36 wks 6 days 5 hr
4. 8 hr 34 min 35 sec
5. 77 deg 50 min 46 sec
6. 12 wks 2 days 19 hr

Mental Calculation

Mental regrouping can be used to avoid the necessity of writing down some of the steps, or of rewriting in columns, when groups of one-digit or two-digit numbers are to be added or subtracted.

One of the most common devices for rapid addition is recognition of groups of digits whose sum is 10. For example, in the following problem two "ten groups" have been marked with braces:

$$\begin{array}{l} 7 \\ \left.\begin{array}{l}6\\4\end{array}\right\} 10 \\ 5 \\ \left.\begin{array}{l}1\\9\end{array}\right\} 10 \\ \hline \end{array}$$

To add this column as grouped, you would say to yourself, "7, 17, 22, 32." The thought should be just the successive totals as shown above and not such cumbersome steps as "7 + 10, 17, + 5, 22, + 10, 32."

When successive digits appear in a column and their sum is less than 10, it is often convenient to think of them, too, as a sum rather than separately. Thus, if adding a column in which the sum of two successive digits is 10 or less, group them as follows:

$$\begin{array}{l} \left.\begin{array}{l}3\\1\\1\end{array}\right\} 5 \\ \left.\begin{array}{l}8\\1\end{array}\right\} 9 \\ \left.\begin{array}{l}4\\6\end{array}\right\} 10 \\ \hline \end{array}$$

The thought process here might be, as shown by the grouping, "5, 14, 24."

Practice problems. Add the following columns from the top down, as in the preceding example:

1.	2.	3.	4.
2	4	88	57
7	6	36	32
3	7	59	64
6	8	82	97
4	1	28	79
1	8	57	44

Answers, showing successive mental steps:

1. 2, 12, 22, 23 - - Final answer, 23
2. 10, 17, 26, 34 - - Final answer, 34
3. Units column: 14, 23, 33, 40 - - Write down 0, carry 4.
   Tens column: 12, 20, 30, 35 - - Final answer, 350.
4. Units column: 9, 20, 29, 33 - - Write down 3, carry 3.
   Tens column: 8, 17, 26, 37 - - Final answer, 373.

SUBTRACTION.—In an example such as 73 - 46, the conventional approach is to place 46 under 73 and subtract units from units and tens from tens, and write only the difference without the intermediate steps. To do this, the best method is to begin at the left. Thus, in the example 73 - 46, we take 40 from 73 and then take 6 from the result. This is done mentally, however, and the thought would be "73, 33, 27," or "33, 27." In the example 84 - 21 the thought is "64, 63" and in the example 64 - 39 the thought is "34, 25."

Practice problems. Mentally subtract and write only the difference:

1. 47 - 24
2. 69 - 38
3. 87 - 58
4. 86 - 73
5. 82 - 41
6. 30 - 12

Answers, showing successive mental steps:

1. 27, 23 - - Final answer, 23
2. 39, 31 - - Final answer, 31
3. 37, 29 - - Final answer, 29
4. 16, 13 - - Final answer, 13
5. 42, 41 - - Final answer, 41
6. 20, 18 - - Final answer, 18

MULTIPLICATION AND DIVISION

Multiplication may be indicated by a multiplication sign (x) between two numbers, a dot between two numbers, or parentheses around one or both of the numbers to be multiplied. The following examples illustrate these methods:

$$6 \times 8 = 48$$
$$6 \cdot 8 = 48$$
$$6(8) = 48$$
$$(6)(8) = 48$$

Notice that when a dot is used to indicate multiplication, it is distinguished from a decimal point or a period by being placed above the line of writing, as in example 2, whereas a period or decimal point appears on the line. Notice also that when parentheses are used to indicate multiplication, the numbers to be multiplied are spaced closer together than they are when the dot or x is used.

In each of the four examples just given, 6 is the MULTIPLIER and 8 is the MULTIPLICAND. Both the 6 and the 8 are FACTORS, and the more modern texts refer to them this way. The "answer" in a multiplication problem is the PRODUCT; in the examples just given, the product is 48.

Division usually is indicated either by a division sign (÷) or by placing one number over another number with a line between the numbers, as in the following examples:

1. $8 \div 4 = 2$

2. $\frac{8}{4} = 2$

The number 8 is the DIVIDEND, 4 is the DIVISOR, and 2 is the QUOTIENT.

MULTIPLICATION METHODS

The multiplication of whole numbers may be thought of as a short process of adding equal numbers. For example, 6(5) and 6 x 5 are read as six 5's. Of course we could write 5 six times and add, but if we learn that the result is 30 we can save time. Although the concept of adding equal numbers is quite adequate in explaining multiplication of whole numbers, it is only a special case of a more general definition, which will be explained later in multiplication involving fractions.

Grouping

Let us examine the process involved in multiplying 6 times 27 to get the product 162. We first arrange the factors in the following manner:

$$\begin{array}{r} 27 \\ \underline{x6} \\ 162 \end{array}$$

The thought process is as follows:

1. 6 times 7 is 42. Write down the 2 and carry the 4.
2. 6 times 2 is 12. Add the 4 that was carried over from step 1 and write the result, 16, beside the 2 that was written in step 1.
3. The final answer is 162.

Table 2-1 shows that the factors were grouped in units, tens, etc. The multiplication was done in three steps: Six times 7 units is 42 units (or 4 tens and 2 units) and six times 2 tens is 12 tens (or 1 hundred and 2 tens). Then the tens were added and the product was written as 162.

Table 2-1.—Multiplying by a one-digit number.

	Hundreds	Tens	Units
		2	7
6(27) = 162			6
		4	2
	1	2	
	1	6	2

In preparing numbers for multiplication as in table 2-1, it is important to place the digits of the factors in the proper columns; that is, units must be placed in the units column, tens in tens column, and hundreds in hundreds column. Notice that it is not necessary to write the zero in the case of 12 tens (120) since the 1 and 2 are written in the proper columns. In practice, the addition is done mentally, and just the product is written without the intervening steps.

Multiplying a number with more than two digits by a one-digit number, as shown in table 2-2, involves no new ideas. Three times 6 units is 18 units (1 ten and 8 units), 3 times 0 tens is 0, and 3 times 4 hundreds is 12 hundreds (1

Table 2-2.—Multiplying a three-digit number by a one-digit number.

	Thousands	Hundreds	Tens	Units
3(406) = 1,218		4	0	6 3
	1	2	1	8
	1	2	1	8

thousand and 2 hundreds). Notice that it is not necessary to write the 0's resulting from the step "3 times 0 tens is 0." The two terminal 0's of the number 1,200 are also omitted, since the 1 and the 2 are placed in their correct columns by the position of the 4.

Partial Products

In the example, 6(8) = 48, notice that the multiplying could be done another way to get the correct product as follows:

$$6(3 + 5) = 6 \times 3 + 6 \times 5$$

That is, we can break 8 into 3 and 5, multiply each of these by the other factor, and add the partial products. This idea is employed in multiplying by a two-digit number. Consider the following example:

```
   43
  x27
1,161
```

Breaking the 27 into 20 + 7, we have 7 units times 43 plus 2 tens times 43, as follows:

$$43(20 + 7) = (43)(7) + (43)(20)$$

Since 7 units times 43 is 301 units, and 2 tens times 43 is 86 tens, we have the following:

```
   43
  x27
  301 = 3 hundreds, 0 tens, 1 unit
  86  = 8 hundreds, 6 tens
1,161
```

As long as the partial products are written in the correct columns, we can multiply beginning from either the left or the right of the multiplier. Thus, multiplying from the left, we have

```
   43
  x27
  86
  301
1,161
```

Multiplication by a number having more places involves no new ideas.

End Zeros

The placement of partial products must be kept in mind when multiplying in problems involving end zeros, as in the following example:

```
   27
  x40
1,080
```

We have 0 units times 27 plus 4 tens times 27, as follows:

```
   27
  x40
    0
 108
1,080
```

The zero in the units place plays an important part in the reading of the final product. End zeros are often called "place holders" since their only function in the problem is to hold the digit positions which they occupy, thus helping to place the other digits in the problem correctly.

The end zero in the foregoing problem can be accounted for very nicely, while at the same time placing the other digits correctly, by means of a shortcut. This consists of offsetting the 40 one place to the right and then simply bringing down the 0, without using it as a multiplier at all. The problem would appear as follows:

$$\begin{array}{r} 27 \\ \underline{x40} \\ 1,080 \end{array}$$

If the problem involves a multiplier with more than one end 0, the multiplier is offset as many places to the right as there are end 0's. For example, consider the following multiplication in which the multiplier, 300, has two end 0's:

$$\begin{array}{r} 220 \\ \underline{x300} \\ 66,000 \end{array}$$

Notice that there are as many place-holding zeros at the end in the product as there are place-holding zeros in the multiplier and the multiplicand combined.

Placement of Decimal Points

In any whole number in the decimal system, there is understood to be a terminating mark, called a decimal point, at the right-hand end of the number. Although the decimal point is seldom shown except in numbers involving decimal fractions (covered in chapter 5 of this course), its location must be known. The placement of the decimal point is automatically taken care of when the end 0's are correctly placed.

Practice problems. Multiply in each of the following problems:

1. 287 x 8	4. 807 x 28
2. 67 x 49	5. 694 x 80
3. 940 x 20	6. 9,241 x 7,800

Answers:

1. 2,296	4. 22,596
2. 3,283	5. 55,520
3. 18,800	6. 72,079,800

DIVISION METHODS

Just as multiplication can be considered as repeated addition, division can be considered as repeated subtraction. For example, if we wish to divide 12 by 4 we may subtract 4 from 12 in successive steps and tally the number of times that the subtraction is performed, as follows:

$$\begin{array}{rl} 12 & \\ \underline{4} & * \\ 8 & \\ \underline{4} & * \\ 4 & \\ \underline{4} & * \\ 0 & \end{array}$$

As indicated by the asterisks used as tally marks, 4 has been subtracted 3 times. This result is sometimes described by saying that "4 is contained in 12 three times."

Since successive subtraction is too cumbersome for rapid, concise calculation, methods which treat division as the inverse of multiplication are more useful. Knowledge of the multiplication tables should lead us to an answer for a problem such as 12 ÷ 4 immediately, since we know that 3 x 4 is 12. However, a problem such as 84 ÷ 4 is not so easy to solve by direct reference to the multiplication table.

One way to divide 84 by 4 is to note that 84 is the same as 80 plus 4. Thus 84 ÷ 4 is the same as 80 ÷ 4 plus 4 ÷ 4. In symbols, this can be indicated as follows:

$$\begin{array}{r} 20 + 1 \\ 4\overline{)80 + 4} \end{array}$$

(When this type of division symbol is used, the quotient is written above the vinculum as shown.) Thus, 84 divided by 4 is 21.

From the foregoing example, it can be seen that the regrouping is useful in division as well as in multiplication. However, the mechanical procedure used in division does not include writing down the regrouped form of the dividend. After becoming familiar with the process, we find that the division can be performed directly, one digit at a time, with the regrouping taking place mentally. The following example illustrates this:

$$\begin{array}{r} 14 \\ 4\overline{)56} \\ \underline{4} \\ 16 \\ \underline{16} \end{array}$$

The thought process is as follows: "4 is contained in 5 once" (write 1 in tens place over the 5); "one times 4 is 4" (write 4 in tens place under 5, take the difference, and bring down 6); and "4 is contained in 16 four times" (write 4 in units place over the 6). After a little practice, many people can do the work shown under the dividend mentally and write only the quotient, if the divisor has only 1 digit.

The divisor is sometimes too large to be contained in the first digit of the dividend. The following example illustrates a problem of this kind:

$$\begin{array}{r} 36 \\ 7\overline{)252} \\ \underline{21} \\ 42 \\ \underline{42} \end{array}$$

Since 2 is not large enough to contain 7, we divide 7 into the number formed by the first two digits, 25. Seven is contained 3 times in 25; we write 3 above the 5 of the dividend. Multiplying, 3 times 7 is 21; we write 21 below the first two digits of the dividend. Subtracting, 25 minus 21 is 4; we write down the 4 and bring down the 2 in the units place of the dividend. We have now formed a new dividend, 42. Seven is contained 6 times in 42; we write 6 above the 2 of the dividend. Multiplying as before, 6 times 7 is 42; we write this product below the dividend 42. Subtracting, we have nothing left and the division is complete.

Estimation

When there are two or more digits in the divisor, it is not always easy to determine the first digit of the quotient. An estimate must be made, and the resulting trial quotient may be too large or too small. For example, if 1,862 is to be divided by 38, we might estimate that 38 is contained 5 times in 186 and the first digit of our trial divisor would be 5. However, multiplication reveals that the product of 5 and 38 is larger than 186. Thus we would change the 5 in our quotient to 4, and the problem would then appear as follows:

$$\begin{array}{r} 49 \\ 38\overline{)1862} \\ \underline{152} \\ 342 \\ \underline{342} \end{array}$$

On the other hand, suppose that we had estimated that 38 is contained in 186 only 3 times. We would then have the following:

$$\begin{array}{r} 3 \\ 38\overline{)1862} \\ \underline{114} \\ 72 \end{array}$$

Now, before we make any further moves in the division process, it should be obvious that something is wrong. If our new dividend is large enough to contain the divisor before bringing down a digit from the original dividend, then the trial quotient should have been larger. In other words, our estimate is too small.

Proficiency in estimating trial quotients is gained through practice and familiarity with number combinations. For example, after a little experience we realize that a close estimate can be made in the foregoing problem by thinking of 38 as "almost 40." It is easy to see that 40 is contained 4 times in 186, since 4 times 40 is 160. Also, since 5 times 40 is 200, we are reasonably certain that 5 is too large for our trial divisor.

Uneven Division

In some division problems such as $7 \div 3$, there is no other whole number that, when multiplied by the divisor, will give the dividend. We use the distributive idea to show how division is done in such a case. For example, $7 \div 3$ could be written as follows:

$$\frac{(6 + 1)}{3} = \frac{6}{3} + \frac{1}{3} = 2\,\frac{1}{3}$$

Thus, we see that the quotient also carries one unit that is to be divided by 3. It should now be clear that $3\overline{)37} = 3\overline{)30 + 7}$, and that this can be further reduced as follows:

$$\frac{30}{3} + \frac{6}{3} + \frac{1}{3} = 10 + 2 + \frac{1}{3} = 12\,\frac{1}{3}$$

In elementary arithmetic the part of the dividend that cannot be divided evenly by the divisor is often called a REMAINDER and is placed next to the quotient with the prefix R. Thus, in the foregoing example where the quotient was $12\,\frac{1}{3}$, the quotient could be written 12 R 1. This method of indicating uneven division is useful in examples such as the following:

Suppose that \$13 is available for the purchase of spare parts, and the parts needed cost \$3 each. Four parts can be bought with the available money, and \$1 will be left over. Since it is not possible to buy 1/3 of a part, expressing the result as 4 R 1 gives a more meaningful answer than 4 1/3.

Placement of Decimal Points

In division, as in multiplication, the placement of the decimal point is important. Determining the location of the decimal point and the number of places in the quotient can be relatively simple if the work is kept in the proper columns. For example, notice the vertical alinement in the following problem:

```
     311
 31/9,641
    9 3
     34
     31
      31
      31
```

We notice that the first two places in the dividend are used to obtain the first place in the quotient. Since 3 is in the hundreds column there are two more places in the quotient (tens place and units place). The decimal point in the quotient is understood to be directly above the position of the decimal point in the dividend. In the example shown here, the decimal point is not shown but is understood to be immediately after the second 1.

Checking Accuracy

The accuracy of a division of numbers can be checked by multiplying the quotient by the divisor and adding the remainder, if any. The result should equal the dividend. Consider the following example:

```
    5203
42/218541      Check:  5203
   210                 x 42
    85                10406
    84               20812
     141             218526
     126               + 15
      15             218541
```

DENOMINATE NUMBERS

We have learned that denominate numbers are not difficult to add and subtract, provided that units, tens, hundreds, etc., are retained in their respective columns. Multiplication and division of denominate numbers may also be performed with comparative ease, by using the experience gained in addition and subtraction.

Multiplication

In multiplying denominate numbers by integers, no new ideas are needed. If in the problem 3(5 yd 2 ft 6 in.) we remember that we can multiply each part separately to get the correct product (as in the example, 6(8) = 6(3) + 6(5)), we can easily find the product, as follows:

```
 5 yd 2 ft  6 in.
             x 3
15 yd 6 ft 18 in.
```

Simplifying, this is

17 yd 1 ft 6 in.

When one denominate number is multiplied by another, a question arises concerning the products of the units of measurement. The product of one unit times another of the same kind is one square unit. For example, 1 ft times 1 ft is 1 square foot, abbreviated sq ft; 2 in. times 3 in. is 6 sq in.; etc. If it becomes necessary to multiply such numbers as 2 yd 1 ft times 6 yd 2 ft, the foot units may be converted to fractions of a yard, as follows:

(2 yd 1 ft)(6 yd 2 ft) = (2 1/3 yd)(6 2/3 yd)

In order to complete the multiplication, a knowledge of fractions is needed. Fractions are discussed in chapter 4 of this training course.

Division

The division of denominate numbers requires division of the highest units first; and if there is a remainder, conversion to the next lower unit, and repeated division until all units have been divided.

In the example (24 gal 1 qt 1 pt) ÷ 5, we perform the following steps:

Step 1:

$$\begin{array}{r} 4 \text{ gal} \\ 5\overline{)24 \text{ gal}} \\ \underline{20} \\ 4 \text{ gal (left over)} \end{array}$$

Step 2: Convert the 4 gal left over to 16 qt and add to the 1 qt.

Step 3:

$$\begin{array}{r} 3 \text{ qt} \\ 5\overline{)17 \text{ qt}} \\ \underline{15} \\ 2 \text{ qt (left over)} \end{array}$$

Step 4: Convert the 2 qt left over to 4 pt and add to the 1 pt.

Step 5:

$$\begin{array}{r} 1 \text{ pt} \\ 5\overline{)5 \text{ pt}} \end{array}$$

Therefore, 24 gal 1 qt 1 pt divided by 5 is 4 gal 3 qt 1 pt.

Practice problems. In problems 1 through 4, divide as indicated. In problems 5 through 8, multiply or divide as indicated.

1. 549 ÷ 9
2. 470/63
3. $25\overline{)2,300}$
4. $64\overline{)74,816}$
5. 4 hr 26 min 16 sec × 5
6. 3(4 gal 3 qt 1 pt)
7. $\dfrac{67 \text{ deg } 43 \text{ min } 12 \text{ sec}}{2}$
8. $5\overline{)63 \text{ lb } 11 \text{ oz}}$

Answers:

1. 61
2. 7 R 29
3. 92
4. 1,169
5. 22 hr 11 min 20 sec
6. 14 gal 2 qt 1 pt
7. 33 deg 51 min 36 sec
8. 12 lb 11 4/5 oz

ORDER OF OPERATIONS

When a series of operations involving addition, subtraction, multiplication, or division is indicated, the order in which the operations are performed is important only if division is involved or if the operations are mixed. A series of individual additions, subtractions, or multiplications may be performed in any order. Thus, in

$$4 + 2 + 7 + 5 = 18$$

or

$$100 - 20 - 10 - 3 = 67$$

or

$$4 \times 2 \times 7 \times 5 = 280$$

the numbers may be combined in any order desired. For example, they may be grouped easily to give

$$6 + 12 = 18$$

and

$$97 - 30 = 67$$

and

$$40 \times 7 = 280$$

A series of divisions should be taken in the order written.

Thus,

$$100 \div 10 \div 2 = 10 \div 2 = 5$$

In a series of mixed operations, perform multiplications and divisions in order from left to right, then perform additions and subtractions in order from left to right.

For example

$$100 \div 4 \times 5 = 25 \times 5 = 125$$

and

$$60 - 25 \div 5 = 60 - 5 = 55$$

Now consider

$$\begin{aligned} 60 - 25 \div 5 + 15 - 100 + 4 \times 10 &= 60 - 5 + 15 - 100 + 4 \times 10 \\ &= 60 - 5 + 15 - 100 + 40 \\ &= 115 - 105 \\ &= 10 \end{aligned}$$

Practice problems. Evaluate each of the following expressions:

1. $9 \div 3 + 2$
2. $18 - 2 \times 5 + 4$
3. $90 \div 2 \div 9$
4. $75 + 5 \times 3 \div 5$
5. $7 + 1 - 8 \times 4 \div 16$

Answers:

1. 5
2. 12
3. 5
4. 9
5. 6

MULTIPLES AND FACTORS

Any number that is exactly divisible by a given number is a MULTIPLE of the given number. For example, 24 is a multiple of 2, 3, 4, 6, 8, and 12, since it is divisible by each of these numbers. Saying that 24 is a multiple of 3, for instance, is equivalent to saying that 3 multiplied by some whole number will give 24. Any number is a multiple of itself and also of 1.

Any number that is a multiple of 2 is an EVEN NUMBER. The even numbers begin with 2 and progress by 2's as follows:

2, 4, 6, 8, 10, 12, ...

Any number that is not a multiple of 2 is an ODD NUMBER. The odd numbers begin with 1 and progress by 2's, as follows:

1, 3, 5, 7, 9, 11, 13, ...

Any number that can be divided into a given number without a remainder is a FACTOR of the given number. The given number is a multiple of any number that is one of its factors. For example, 2, 3, 4, 6, 8, and 12 are factors of 24. The following four equalities show various combinations of the factors of 24:

$$24 = 24 \cdot 1 \qquad 24 = 8 \cdot 3$$
$$24 = 12 \cdot 2 \qquad 24 = 6 \cdot 4$$

If the number 24 is factored as completely as possible, it assumes the form

$$24 = 2 \cdot 2 \cdot 2 \cdot 3$$

ZERO AS A FACTOR

If any number is multiplied by zero, the product is zero. For example, 5 times zero equals zero and may be written $5(0) = 0$. The zero factor law tells us that, if the product of two or more factors is zero, at least one of the factors must be zero.

PRIME FACTORS

A number that has factors other than itself and 1 is a COMPOSITE NUMBER. For example, the number 15 is composite. It has the factors 5 and 3.

A number that has no factors except itself and 1 is a PRIME NUMBER. Since it is sometimes advantageous to separate a composite number into prime factors, it is helpful to be able to recognize a few prime numbers quickly. The following series shows all the prime numbers up to 60:

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59.

Notice that 2 is the only even prime number. All other even numbers are divisible by 2. Notice also that 51, for example, does not appear in the series, since it is a composite number equal to 3×17.

If a factor of a number is prime, it is called a PRIME FACTOR. To separate a number into prime factors, begin by taking out the smallest factor. If the number is even, take out all the 2's first, then try 3 as a factor, etc. Thus, we have the following example:

$$\begin{aligned} 540 &= 2 \cdot 270 \\ &= 2 \cdot 2 \cdot 135 \\ &= 2 \cdot 2 \cdot 3 \cdot 45 \\ &= 2 \cdot 2 \cdot 3 \cdot 3 \cdot 15 \\ &= 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 \cdot 5 \end{aligned}$$

Since 1 is an understood factor of every number, we do not waste space recording it as one of the factors in a presentation of this kind.

A convenient way of keeping track of the prime factors is in the short division process as follows:

$$\begin{array}{r|l} 2 & 540 \\ 2 & 270 \\ 3 & 135 \\ 3 & 45 \\ 3 & 15 \\ 5 & 5 \\ & 1 \end{array}$$

If a number is odd, its factors will be odd numbers. To separate an odd number into prime factors, take out the 3's first, if there are any. Then try 5 as a factor, etc. As an example,

$$\begin{aligned} 5{,}775 &= 3 \cdot 1{,}925 \\ &= 3 \cdot 5 \cdot 385 \\ &= 3 \cdot 5 \cdot 5 \cdot 77 \\ &= 3 \cdot 5 \cdot 5 \cdot 7 \cdot 11 \end{aligned}$$

Practice problems:

1. Which of the following are prime numbers and which are composite numbers?

   25, 7, 18, 29, 51

2. What prime numbers are factors of 36?

3. Which of the following are multiples of 3?

   45, 53, 51, 39, 47

4. Find the prime factors of 27.

Answers:

1. Prime: 7, 29
   Composite: 25, 18, 51
2. $36 = 2 \cdot 2 \cdot 3 \cdot 3$
3. 45, 51, 39
4. $27 = 3 \cdot 3 \cdot 3$

Tests for Divisibility

It is often useful to be able to tell by inspection whether a number is exactly divisible by one or more of the digits from 2 through 9. An expression which is frequently used, although it is sometimes misleading, is "evenly divisible." This expression has nothing to do with the concept of even and odd numbers, and it probably should be avoided in favor of the more descriptive expression, "exactly divisible." For the remainder of this discussion, the word "divisible" has the same meaning as "exactly divisible." Several tests for divisibility are listed in the following paragraphs:

1. A number is divisible by 2 if its right-hand digit is even.
2. A number is divisible by 3 if the sum of its digits is divisible by 3. For example, the digits of the number 6,561 add to produce the sum 18. Since 18 is divisible by 3, we know that 6,561 is divisible by 3.
3. A number is divisible by 4 if the number formed by the two right-hand digits is divisible by 4. For example, the two right-hand digits of the number 3,524 form the number 24. Since 24 is divisible by 4, we know that 3,524 is divisible by 4.
4. A number is divisible by 5 if its right-hand digit is 0 or 5.
5. A number is divisible by 6 if it is even and the sum of its digits is divisible by 3. For example, the sum of the digits of 64,236 is 21, which is divisible by 3. Since 64,236 is also an even number, we know that it is divisible by 6.
6. No short method has been found for determining whether a number is divisible by 7.
7. A number is divisible by 8 if the number formed by the three right-hand digits is divisible by 8. For example, the three right-hand digits of the number 54,272 form the number 272, which is divisible by 8. Therefore, we know that 54,272 is divisible by 8.
8. A number is divisible by 9 if the sum of its digits is divisible by 9. For example, the sum of the digits of 546,372 is 27, which is divisible by 9. Therefore we know that 546,372 is divisible by 9.

Practice problems. Check each of the following numbers for divisibility by all of the digits except 7:

1. 242,431,231,320
2. 844,624,221,840
3. 988,446,662,640
4. 207,634,542,480

Answers: All of these numbers are divisible by 2, 3, 4, 5, 6, 8, and 9.